

# Coherent transport in disordered metals out of equilibrium

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**Abstract.** We derive a formula for the quantum corrections to the electrical current for a metal out of equilibrium. In the limit of linear current-voltage characteristics our formula reproduces the well known Altshuler-Aronov correction to the conductivity of a disordered metal. The current formula is obtained by a direct diagrammatic approach, and is shown to agree with what is obtained within the Keldysh formulation of the non-linear sigma model. As an application we calculate the current of a mesoscopic wire. We find a current-voltage characteristics that scales with  $eV/kT$ , and calculate the different scaling curves for a wire in the hot-electron regime and in the regime of full non-equilibrium.

**PACS.** 72.10.-d Theory of electronic transport; scattering mechanisms – 72.15.Rn Localization effects (Anderson or weak localization)

## 1 Introduction

Quantum interference effects in disordered metals have been the subject of intensive investigation for over twenty years. For general reviews see [1–3]. The interference of the scattered electrical waves in the presence of a random potential leads to corrections [4–6] to the semi-classical formula of the electrical conductivity known from the Drude-Boltzmann theory. The physical implications of these quantum corrections have been extensively discussed in the literature mostly for the equilibrium properties, for which experimental data were available, although a number of non-linear electric field effects have been predicted in the past [2, 7–10].

In contrast, non-equilibrium electrical transport has received considerable attention in the field of mesoscopic physics. Examples are the transport in quantum dots [11], or the shot noise in mesoscopic conductors [12]. In this situation however the majority of the studied phenomena did not involve interference effects.

Our interest in the non-equilibrium properties of interference phenomena originated by the suggestion [13] that the non-equilibrium electric noise could be the origin for the low temperature saturation of the weak localization dephasing time observed in disordered films and wires [14]. Mohanty *et al.* [15] pointed out that in the samples with the strongest dephasing rate also the interaction correction (Altshuler-Aronov) to the conductivity saturates at low temperature. This suggests that also the Altshuler-Aronov correction should be affected by the non-equilibrium noise. Similar speculations concerning

an electric field effect on the Altshuler-Aronov correction have also been made earlier by different authors in different contexts [16, 17].

This motivated us to study the interaction correction in the presence of an external field. In analogy to the analysis performed in the literature for weak localization [2], we calculated the interaction correction to the current in the presence of a time dependent vector potential, assuming a thermal distribution function [18]. Indeed we verified the above-mentioned speculations since we found an electric field effect. However, contrary to what is known in the case of weak localization, where the strongest effect occurs when the period of the AC field is of the order of the dephasing time, we found a suppression of the interaction correction even by static electric fields. These findings have however raised the issues of the possibility of experimentally observing the effect and of its physical interpretation. Both these questions will be addressed in this paper.

Let us comment first the problem concerning the interpretation of the effect. In reference [19] we demonstrated that the non-linear field effect can be understood in terms of dephasing by calculating the phase shifts of the relevant classical paths in the presence of a time dependent vector potential. On the other hand, a static electric field can also be described in terms of a static scalar potential, and it is clear that a static scalar potential does not lead to dephasing. Therefore we think it is of interest to present a version of our theory in the scalar gauge. In this paper we will explicitly show the gauge invariance of our previous results.

The second problem concerns the scale of the effect and its experimental observability. We found in reference [18]

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that the temperature-dependent Altshuler-Aronov correction to the conductivity saturates when the voltage drop on the thermal length ( $L_T = \sqrt{D/T}$ ) is comparable to the temperature  $eEL_T \sim kT$ . From this condition one can estimate the strength of the microwave field that is necessary to explain the saturation of the Altshuler-Aronov correction observed in the experimental data of reference [15]. In so doing one arrives at a microwave field value that is more than an order of magnitude larger than the optimistic estimate given in reference [13] to explain the saturation of the weak localization time. Furthermore it is rather unlikely that the condition  $eEL_T \sim kT$  can be reached at low temperature, since strong heating is already assumed to set in when the voltage drop over the electron-phonon length is of order of the temperature. Whether the experimentally observed saturation of the resistance [15] is due to heating or not is hard to decide since the electron temperature has not been measured directly. Similar problems arise also in the attempt to explain the experimental data of references [16, 17].

Despite the above mentioned problems, we think we cannot rule out non-equilibrium noise as a reason for the observed saturations. In fact, even in the absence of strong heating, the distribution function may deviate from the equilibrium form and affect the interaction correction to the conductivity and possibly lead to saturation at considerably weaker electric fields. A theory which is valid even out of equilibrium will be developed in this paper.

In contrast to our previous work [18] we will avoid to guess the distribution function which could be relevant for the experiments of references [14, 15]. Instead we will calculate the interaction correction in a more controlled situation. Nowadays it is, indeed, possible to create non-equilibrium in a controlled way by, for instance, attaching a short mesoscopic wire to large metallic reservoirs (see *e.g.* [20]). In the absence of inelastic scattering processes the distribution function in the wire is a linear superposition of the distribution functions of the leads [21], at least when only the ensemble averaged distribution function is considered as it is the case here, see also reference [22] for comparison. The interaction correction in such a situation has also been considered by Nagaev [10] at zero temperature and in the recent paper by Gutman and Gefen [23] for finite temperature. We restrict ourselves to nonzero temperature where we verify the result that the  $I - V$  characteristics scales as  $eV/kT$ . Going beyond the analysis by Gutman and Gefen, we calculate the  $I - V$  characteristics explicitly and we will compare quantitatively the wire in non-equilibrium with the wire in the hot electron regime. In addition we will also discuss the interaction correction in the spin triplet channels.

Our paper is organized as follows. In the next section we introduce the basic quantities and recall the main results of the Drude-Boltzmann theory within the Keldysh formalism. In Section 3 we consider the quantum corrections to the conductivity within the Keldysh diagrammatic approach. We derive, in particular, an expression for the current in the presence of an external electric field. In Section 4 we discuss the gauge invariance of the the-

ory, while in Section 5 we present a specific application: a mesoscopic wire. Finally in Section 6 we give our conclusions. In the appendices we outline how to obtain the same results using the Keldysh formulation of the non-linear sigma model and we extend the calculations in order to include also the spin effects.

## 2 Basic definitions and the Drude-Boltzmann theory

In this section we will recall some basic relations of the quasi-classical approximation in its non-equilibrium (Keldysh) formulation [24]. Our notation will mainly follow reference [25]. We will write down the equation of motion for the Green functions in the presence of impurity scattering in the case when quantum interference is completely neglected. The Green functions have the matrix structure

$$\hat{G} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix}, \quad (1)$$

with

$$G^R(x, x') = -i\Theta(t - t') (\langle \Psi(x)\Psi^\dagger(x') + \Psi^\dagger(x')\Psi(x) \rangle) \quad (2)$$

$$G^A(x, x') = +i\Theta(t' - t) (\langle \Psi(x)\Psi^\dagger(x') + \Psi^\dagger(x')\Psi(x) \rangle) \quad (3)$$

$$G^K(x, x') = -i (\langle \Psi(x)\Psi^\dagger(x') - \Psi^\dagger(x')\Psi(x) \rangle), \quad (4)$$

where  $\Psi$  and  $\Psi^\dagger$  are fermion operators and  $x = (\mathbf{x}, t)$ . In equilibrium the Keldysh component of the Green function is expressed in terms of the retarded and advanced components by  $G_\epsilon^K = [1 - 2f(\epsilon)](G_\epsilon^R - G_\epsilon^A)$ , where  $f(\epsilon)$  is the Fermi function. The Keldysh component out of equilibrium will be discussed later.

The Green function solves the differential equation

$$\left( i\frac{\partial}{\partial t} + \frac{1}{2m}(\nabla + ie\mathbf{A})^2 + e\phi + \mu \right) \hat{G}(\mathbf{x}, t; \mathbf{x}', t') - \int dt_1 d\mathbf{x}_1 \hat{\Sigma}(\mathbf{x}, t; \mathbf{x}_1, t_1) \hat{G}(\mathbf{x}_1, t_1; \mathbf{x}', t') = \delta(\mathbf{x} - \mathbf{x}')\delta(t - t'), \quad (5)$$

where  $\phi$  and  $\mathbf{A}$  are the scalar and the vector potential. Since the self-energy  $\hat{\Sigma}$  has the same triangular matrix structure as the Green function, one can invert the inverse Green function  $\hat{G}^{-1}$  and finds for the Keldysh component the relation

$$G^K = G^R \Sigma^K G^A. \quad (6)$$

For a graphical representation see Figure 1. We then introduce the  $\xi$ -integrated (quasi-classical) Green function

$$\hat{g}_{tt'}(\hat{\mathbf{p}}, \mathbf{R}) = \frac{i}{\pi} \int d\xi d\mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} \hat{G}\left(\mathbf{R} + \frac{\mathbf{r}}{2}, t; \mathbf{R} - \frac{\mathbf{r}}{2}, t'\right), \quad (7)$$

$$\begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix} = \begin{pmatrix} \leftarrow & \leftarrow \boxed{\phantom{x}} \leftarrow \\ 0 & \leftarrow \leftarrow \end{pmatrix}$$

**Fig. 1.** Graphical representation of the Green function; the shaded box in  $G^K$  represents the Keldysh component of the self-energy, *i.e.*, basically the distribution function.

where  $\xi = \mathbf{p}^2/2m - \mu$  and  $\hat{\mathbf{p}}$  is a unit vector along the momentum. The Green function in the energy domain is

$$\hat{g}_{\epsilon\epsilon'}(\hat{\mathbf{p}}, \mathbf{x}) = \int dt dt' e^{i\epsilon t - i\epsilon' t'} \hat{g}_{tt'}(\hat{\mathbf{p}}, \mathbf{x}). \quad (8)$$

We will keep the notation of small  $g$  for the  $\xi$ -integrated Green functions and capital  $G$  for the not integrated Green functions all over this paper. When approximating the density of states as an energy independent constant, the  $\xi$ -integration is related to an integration over the momentum  $\mathbf{p}$  according to

$$\int \frac{d^3 p}{(2\pi)^3} \rightarrow N_0 \int d\xi \int \frac{d\hat{p}}{4\pi}. \quad (9)$$

We will now recall some relations that are specific for impurity scattering. By treating the impurity scattering within the self-consistent Born approximation and assuming a Gaussian,  $\delta$ -correlated impurity potential with

$$\langle U(\mathbf{x})U(\mathbf{x}') \rangle = \frac{1}{2\pi N_0 \tau} \delta(\mathbf{x} - \mathbf{x}'), \quad (10)$$

the electron self-energy is local in space and is given by

$$\hat{\Sigma}^{\text{imp}}(\mathbf{x}, t; \mathbf{x}', t') = \frac{1}{2\pi N_0 \tau} \hat{G}(\mathbf{x}, t; \mathbf{x}, t') \delta(\mathbf{x} - \mathbf{x}') \quad (11)$$

$$= \hat{\Sigma}_{tt'}^{\text{imp}}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}'). \quad (12)$$

Notice that this equation has to be solved self-consistently for all the components of the Green function. Using the above definition, one observes that the impurity self-energy is related to the  $s$ -wave part of the quasi-classical function,

$$\hat{\Sigma}_{tt'}^{\text{imp}}(\mathbf{x}) = -\frac{i}{2\tau} \int \frac{d\hat{p}}{4\pi} \hat{g}_{tt'}(\hat{\mathbf{p}}, \mathbf{x}). \quad (13)$$

The distribution function out of equilibrium is found by solving the appropriate kinetic equation. Here we derive the kinetic equation for  $g_{\epsilon\epsilon'}^K$ , from equations (6) and (7). For simplicity we neglect external fields for the time being. Under these conditions the retarded and advanced Green functions are

$$G^{\text{R(A)}}(\mathbf{p}, \epsilon) = \frac{1}{\epsilon - \xi \pm i/2\tau}. \quad (14)$$

Near the Fermi energy ( $\epsilon, \epsilon' \ll \epsilon_F$ ) and for small momenta ( $q \ll p_F$ ) one finds

$$g_{\epsilon\epsilon'}^K(\hat{\mathbf{p}}, \mathbf{q}) = \frac{i}{\pi} \int d\xi G^{\text{R}}(\epsilon, \mathbf{p} + \mathbf{q}/2) \times \Sigma_{\epsilon\epsilon'}^K(\mathbf{q}) G^{\text{A}}(\epsilon', \mathbf{p} - \mathbf{q}/2) \quad (15)$$

$$\approx \frac{i}{\tau} \frac{1}{\epsilon - \epsilon' + i/\tau - v_F \hat{\mathbf{p}} \cdot \mathbf{q}} \int \frac{d\hat{p}}{4\pi} g_{\epsilon\epsilon'}^K(\hat{\mathbf{p}}, \mathbf{q}). \quad (16)$$

The equation above reproduces the well-known kinetic equation for impurity scattering

$$\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial t'} + v_F \hat{\mathbf{p}} \cdot \nabla \right) g_{tt'}^K(\hat{\mathbf{p}}, \mathbf{x}) = \frac{1}{\tau} \left( g_{tt'}^K(\hat{\mathbf{p}}, \mathbf{x}) - \int \frac{d\hat{p}}{4\pi} g_{tt'}^K(\hat{\mathbf{p}}, \mathbf{x}) \right). \quad (17)$$

In this work we will restrict to the case, where energies and momenta are restricted even more, namely  $\epsilon\tau, \epsilon'\tau, qv_F\tau \ll 1$ . By expanding (16) for small energy and momentum and taking the angular average, one finds the diffusive equation

$$\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial t'} - D \frac{\partial^2}{\partial \mathbf{x}^2} \right) \int \frac{d\hat{p}}{4\pi} g_{tt'}^K(\hat{\mathbf{p}}, \mathbf{x}) = 0, \quad (18)$$

where the diffusion constant is  $D = v_F^2 \tau / 3$ . Notice that this equation is solved by any function  $g_{tt'}^K(\mathbf{p}, \mathbf{x})$  which is independent of position  $\mathbf{x}$  and which depends on time differences ( $t - t'$ ) only. This reflects the fact that any distribution function is allowed for noninteracting electrons.

The charge density and current density are related to the Keldysh component of the Green function,

$$\rho(\mathbf{x}, t) = ie G^K(\mathbf{x}, t; \mathbf{x}, t) \quad (19)$$

$$\mathbf{j}(\mathbf{x}, t) = \frac{e}{2m} [\nabla_{\mathbf{x}} - \nabla_{\mathbf{x}'} + 2ieA(\mathbf{x}, t)] G^K(\mathbf{x}, t; \mathbf{x}', t)|_{\mathbf{x}'=\mathbf{x}}. \quad (20)$$

In terms of the quasi-classical Green functions, the charge and current read [25]

$$\rho(\mathbf{x}, t) = 2eN_0 \left( \frac{\pi}{2} \int \frac{d\hat{p}}{4\pi} g_{tt}^K(\hat{\mathbf{p}}, \mathbf{x}) - e\phi(\mathbf{x}, t) \right) \quad (21)$$

$$\mathbf{j}(\mathbf{x}, t) = e\pi N_0 \int \frac{d\hat{p}}{4\pi} v_F \hat{\mathbf{p}} g_{tt}^K(\hat{\mathbf{p}}, \mathbf{x}). \quad (22)$$

It is useful to consider the current density in the presence of an electric field  $\mathbf{E}(\mathbf{x}) = -\nabla\phi(\mathbf{x})$ . By replacing  $G^K$  in (20) with  $G^R \Sigma^K G^A$  we express the current density as

$$\mathbf{j}(\mathbf{q}, \omega) = ie \int \frac{d\epsilon}{2\pi} \int \frac{d^3 p}{(2\pi)^3} \frac{\mathbf{p}}{m} G^{\text{R}} \left( \epsilon + \frac{\omega}{2}, \mathbf{p} + \frac{\mathbf{q}}{2} \right) \times \Sigma_{\epsilon+\omega/2, \epsilon-\omega/2}^K(\mathbf{q}) G^{\text{A}} \left( \epsilon - \frac{\omega}{2}, \mathbf{p} - \frac{\mathbf{q}}{2} \right), \quad (23)$$

from which we obtain

$$\mathbf{j}(\mathbf{x}, t) = -e\pi D N_0 \nabla \int \frac{d\hat{p}}{4\pi} g_{tt}^K(\hat{\mathbf{p}}, \mathbf{x}) \quad (24)$$

$$= -D \nabla \rho(\mathbf{x}, t) + 2e^2 D N_0 \mathbf{E}(\mathbf{x}, t). \quad (25)$$

Within the adopted approximations, *i.e.*, a constant density of states and a uniform diffusion coefficient, one observes that the current is a linear function of the electric field as long as the charge density  $\rho(\mathbf{x})$  stays uniform.

We close this section by commenting on equation (21). A scalar field  $\phi(\mathbf{x}, t)$  shifts the entire Fermi surface, *i.e.*, it affects the Green function at all energies. This is lost in the naive substitution of equation (9), when the  $\xi$ -integration is extended to  $\pm\infty$ . The second term in equation (21) is obtained by taking into account these high energy terms correctly. The equilibrium response to a static field is, for instance, fully given by this second contribution. For this reason the second term is often referred to as the “static contribution” to the response, whereas the first term is referred to as the “dynamic contribution”.

### 3 Quantum correction to the current

Quantum interference gives rise to corrections to the semi-classical expression of the electrical conductivity of a metal. The so-called quantum corrections to the average conductivity are the weak localization correction (WL), the interaction correction in the particle-hole channel (EEI), and the interaction correction in the Cooper channel (EEIC). In this paper we will concentrate on the interaction correction in the particle-hole channel. For non-linear effects in WL we refer to the literature [2,3]. Interactions in the Cooper channel will not be considered. This is justified for non-super-conducting materials since in that situation the relevant interaction parameter scales downwards under the renormalization group.

#### 3.1 Ladder diagrams

Before calculating the quantum corrections we introduce the ladder diagrams of repeated impurity scattering which will appear at various places in the diagrammatic approach. Technically speaking these ladder diagrams appear when averaging a product of a retarded and an advanced Green function. Here we briefly recall how to derive the expressions for the ladder in the absence of external fields and without spin effects. The inclusion of external fields and spin structure is straightforward and one may refer to the reviews on the subject like reference [2].

The diffuson  $D(\mathbf{q}, \omega)$  or particle-hole ladder is found by summing the sequence of diagrams shown in Figure 2:

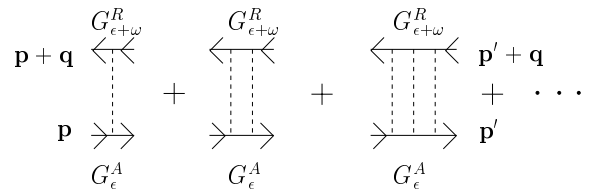
$$D(\mathbf{q}, \omega) = 1 + \eta^{\text{RA}} + (\eta^{\text{RA}})^2 + \dots = \frac{1}{1 - \eta^{\text{RA}}} \quad (26)$$

with

$$\begin{aligned} \eta^{\text{RA}} &= \frac{1}{2\pi N_0 \tau} \int \frac{d^3 p}{(2\pi)^3} G^{\text{R}}(\epsilon + \omega, \mathbf{p} + \mathbf{q}) G^{\text{A}}(\epsilon, \mathbf{p}) \quad (27) \\ &\approx 1 - \tau(-i\omega + Dq^2) \quad (28) \end{aligned}$$

where we have used the condition that  $\omega\tau \ll 1$  and  $v_{\text{F}}q\tau \ll 1$  so that the diffuson reads

$$D(\mathbf{q}, \omega) = \frac{1}{\tau - i\omega + Dq^2}. \quad (29)$$



**Fig. 2.** Graphical definition of the diffuson (particle-hole ladder).  $G^{\text{R}}$  and  $G^{\text{A}}$  are the retarded and advanced Green functions, which in the general case can depend on external electromagnetic fields.

$$D_{tt'}^{\eta}(\mathbf{x}, \mathbf{x}') = \begin{array}{c} \begin{array}{ccc} & G^{\text{R}} & \\ \mathbf{x}, t + \eta/2 & \leftarrow & \mathbf{x}', t' + \eta/2 \\ & \vdots & \\ \mathbf{x}, t - \eta/2 & \rightarrow & \mathbf{x}', t' - \eta/2 \\ & G^{\text{A}} & \end{array} \end{array}$$

**Fig. 3.** The diffuson in the space/time domain.

For completeness we give the expression in the presence of external electromagnetic fields. In this case it is convenient to go a real space representation where the diffuson is defined by the equation

$$\left\{ \frac{\partial}{\partial t} - D(\nabla_{\mathbf{x}} + ie\mathbf{A}_D)^2 - ie\phi_D \right\} D_{tt'}^{\eta}(\mathbf{x}, \mathbf{x}') = \frac{1}{\tau} \delta(\mathbf{x} - \mathbf{x}') \delta(t - t'), \quad (30)$$

with  $\mathbf{A}_D = \mathbf{A}(\mathbf{x}, t + \eta/2) - \mathbf{A}(\mathbf{x}, t - \eta/2)$  and  $\phi_D = \phi(\mathbf{x}, t + \eta/2) - \phi(\mathbf{x}, t - \eta/2)$ . In these equations  $t$  is the center-of-mass time, and  $\eta$  is the relative time, and are defined in Figure 3. Notice that the external field drops from the equation for the diffuson when the relative time  $\eta$  equals zero.

#### 3.2 Interaction correction to the current

We are now ready to allow for electron-electron interactions. Interactions will enter the kinetic equation and determine the form of the distribution function. We assume that the distribution function has been determined self-consistently *via* the kinetic equation with the inclusion of the interaction. We will concentrate then on the calculation of the interaction corrections to the current density. To do so we need the expression for the Keldysh Green function in the presence of interactions. Following references [26,25] we start with the self-energy

$$\Sigma = \Sigma^{\text{imp}} + \Sigma^{\text{V}} \quad (31)$$

where  $\Sigma^{\text{imp}}$  is the previously defined impurity self-energy and

$$\begin{aligned} \Sigma_{ij}^{\text{V}}(x, x') &= i \sum_{i'j'kk'} \int dx_2 dx_3 dx_4 dx_5 \Gamma_{ii'}^k(x_5; x, x_3) \\ &\quad \times V^{kk'}(x_5, x_4) G_{i'j'}(x_3, x_2) \tilde{\Gamma}_{jj'}^{k'}(x_4; x_2, x'). \end{aligned} \quad (32)$$

**Fig. 4.** The self-energy containing both disorder and interaction.

**Fig. 5.** Dressing of the interaction vertex with impurity lines.

The vertex functions are given by

$$\Gamma_{ij}^k(x; x_1, x_2) = \gamma_{ij}^k + \frac{1}{2\pi N_0 \tau} \sum_{i'j'} \int dx'_1 dx'_2 G_{ii'}(x_1, x'_1) \times \Gamma_{i'j'}^k(x; x'_1, x'_2) G_{j'j}(x'_2, x_2). \quad (33)$$

An analogous equation is valid for  $\tilde{\Gamma}$ . We recall that in the Keldysh triangular representation the “absorption” and “emission” vertices differ. A diagrammatic representation of both the self-energy and vertex equations is shown in Figures 4 and 5.

The indices  $i, j, \dots$  denote matrix indices in Keldysh space. The bare vertices  $\gamma, \tilde{\gamma}$  are local in space and time. The structure in Keldysh space is  $\gamma_{ij}^1 = \tilde{\gamma}_{ij}^2 = \delta_{ij}/\sqrt{2}$  and  $\gamma_{ij}^2 = \tilde{\gamma}_{ij}^1 = \sigma_{ij}^x/\sqrt{2}$ . From  $\Sigma^{\text{imp}} + \Sigma^V$  one may derive a kinetic equation for the system with disorder and interaction. The general expressions were already given in the seminal paper, reference [26]. Explicit expressions for the various components of  $\Gamma$  and  $\tilde{\Gamma}$  in terms of integrals like  $\eta^{\text{RA}}$  are also given in the appendix of reference [25]. Fully *evaluating* the expressions in this or that limit remains still to be done. Even in thermal equilibrium we are not aware of any full self-consistent calculation.

In the following we will take into account only the non-interacting self-energy self-consistently, and restrict ourselves to the perturbation theory for the interacting part of the problem. The change in  $G^{\text{K}}$  due to the interaction may be written as

$$\delta G^{\text{K}} = G^{\text{R}} \delta \Sigma^{\text{R}} G^{\text{K}} + G^{\text{R}} \delta \Sigma^{\text{K}} G^{\text{A}} + G^{\text{K}} \delta \Sigma^{\text{A}} G^{\text{A}}, \quad (34)$$

where  $\delta \Sigma$  is a sum of the interaction self-energy plus the interaction-induced change in the impurity self-energy

$$\delta \Sigma = \delta \Sigma^{\text{imp}} + \Sigma^V. \quad (35)$$

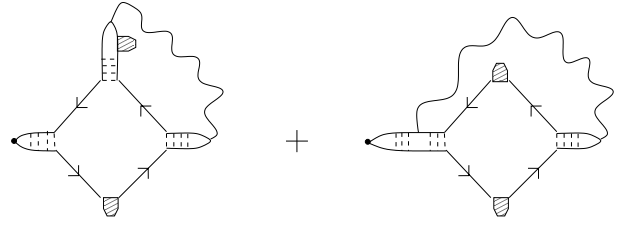
Among the many contributions to  $\delta \Sigma$  we start with the Keldysh component of  $\delta \Sigma^{\text{imp}}$ . We denote the corresponding correction to the current as  $\delta \mathbf{j}_a$ , which we determine as

$$\delta \mathbf{j}_a(\mathbf{x}, t) = -ieD2\pi N_0 \tau \nabla \delta \Sigma^{\text{imp}, \text{K}}. \quad (36)$$

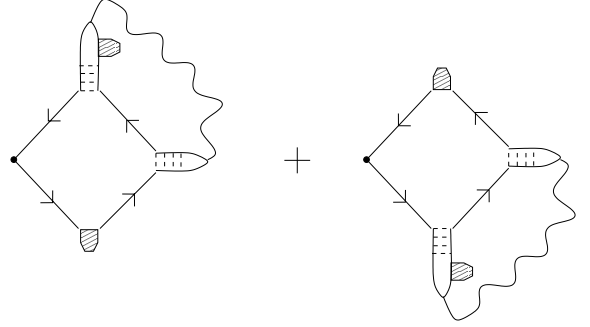
Apparently  $\delta \mathbf{j}_a$  is related to the correction to the charge density and may be written as

$$\delta \mathbf{j}_a(\mathbf{x}, t) = -e\pi DN_0 \nabla \int \frac{d\hat{p}}{4\pi} \delta g_{tt}^{\text{K}}(\mathbf{p}, \mathbf{x}) \quad (37)$$

$$= -D \nabla \delta \rho(\mathbf{x}). \quad (38)$$



**Fig. 6.** Interaction correction to the current; diagrams of the type shown here correspond to  $\delta \mathbf{j}_a$  and may be related to the gradient of the density.



**Fig. 7.** Interaction correction to the current; these diagrams contribute to  $\delta \mathbf{j}_b$ .

Some typical diagrams contributing to  $\delta \mathbf{j}_a$  are shown in Figure 6. The calculation of the correction to the current from all the other contributions, which we will denote by  $\delta \mathbf{j}_b$ , simplifies due to the following observation: the majority of the components of the renormalized vertices  $\Gamma, \tilde{\Gamma}$  are strongly enhanced over the bare value due to the presence of a diffusive type of vertex correction. The explicit calculation [25] shows, however, that the diagonal parts of  $\Gamma_{ij}^1$  and  $\tilde{\Gamma}_{ij}^2$  are not renormalized. In the leading order of a gradient expansion we can therefore neglect all the terms involving these vertices. As a result the expression for the various self-energy components are given by the combinations [25]

$$\Sigma^{\text{V}, \text{R}} : \Gamma_{12}^1 G_{22} \tilde{\Gamma}_{21}^1 V^{\text{R}} \quad (39)$$

$$\Sigma^{\text{V}, \text{A}} : \Gamma_{21}^2 G_{11} \tilde{\Gamma}_{12}^2 V^{\text{A}} \quad (40)$$

$$\Sigma^{\text{V}, \text{K}} : \Gamma_{12}^1 G_{22} \tilde{\Gamma}_{22}^1 V^{\text{R}} + \Gamma_{11}^2 G_{11} \tilde{\Gamma}_{12}^2 V^{\text{A}} \quad (41)$$

and

$$\delta \Sigma^{\text{imp}, \text{R}} : G^{\text{R}} \Sigma^{\text{V}, \text{R}} G^{\text{R}} / (2\pi N_0 \tau) \quad (42)$$

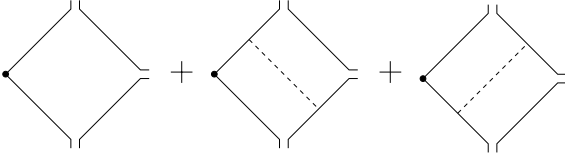
$$\delta \Sigma^{\text{imp}, \text{A}} : G^{\text{A}} \Sigma^{\text{V}, \text{A}} G^{\text{A}} / (2\pi N_0 \tau). \quad (43)$$

A diagrammatic representation of the correction to the current from  $\Sigma^{\text{V}, \text{R}}$  and  $\Sigma^{\text{V}, \text{A}}$  is shown in Figure 7. Going through the algebra one convinces one-self that the contributions from  $\Sigma^{\text{V}}$  and  $\delta \Sigma^{\text{imp}}$  may be combined as

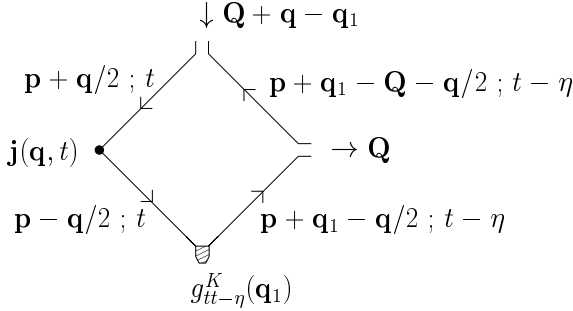
$$\Sigma^{\text{V}, \text{R}} + \delta \Sigma^{\text{imp}, \text{R}} + \Sigma_a^{\text{V}, \text{K}} \quad (44)$$

$$\Sigma^{\text{V}, \text{A}} + \delta \Sigma^{\text{imp}, \text{A}} + \Sigma_b^{\text{V}, \text{K}} \quad (45)$$

where  $\Sigma_a^{\text{V}, \text{K}}$  and  $\Sigma_b^{\text{V}, \text{K}}$  refer to the two terms entering the expression for  $\Sigma^{\text{V}, \text{K}}$ . These combinations of terms are



**Fig. 8.** The three diagrams constituting the Hikami box; within the here-applied formalism the diagrams are generated from the self-energies  $\delta\Sigma^{\text{imp}} + \Sigma^V$ .



**Fig. 9.** Labeling of momenta and times in the Hikami box; within the here-applied approximations the retarded and advanced Green functions are local in time.

taken into account in the diagrams of Figure 7 by completing the “Hikami box” as shown in Figure 8. We can now proceed to the explicit calculation of the correction to the current. Let us start with the evaluation of the Hikami box. The first of the diagrams of the Hikami box is shown in more detail in Figure 9. The evaluation of the diagram amounts to the integration over the “fast” momentum entering the electron Green function and is accomplished by

$$\int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{m} G^R(\mathbf{p} + \mathbf{q}/2) G^A(\mathbf{p} - \mathbf{q}/2) \times G^R(\mathbf{p} - \mathbf{q}/2 + \mathbf{q}_1) G^A(\mathbf{p} - \mathbf{q}/2 + \mathbf{q}_1 - \mathbf{Q}). \quad (46)$$

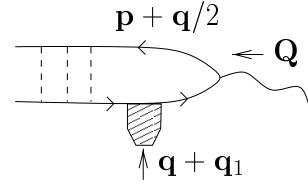
The integration is performed under the assumption that the three momenta  $\mathbf{q}$ ,  $\mathbf{q}_1$ , and  $\mathbf{Q}$  are small compared to  $1/l$  and that the energy in all the four Green functions is small compared to  $1/\tau$ . After expanding the Green functions to first order in  $\mathbf{q}$ ,  $\mathbf{q}_1$ ,  $\mathbf{Q}$ , the integration over  $\mathbf{p}$  gives  $(-6\pi i N_0 \tau^2) D(\mathbf{Q} + \mathbf{q})$ . Evaluating the other two diagrams of Figure 8 analogously and summing up the three of them one arrives at  $(-4\pi i N_0 \tau^3) D\mathbf{Q}$ .

After completing the evaluation of the Hikami box, we now address our attention to the vertex functions  $\Gamma$  and  $\tilde{\Gamma}$ . From reference [25] we borrow the relevant expressions as

$$\Gamma_{12}^1 = \frac{1}{\sqrt{2}} (1 - \eta^{\text{RA}})^{-1} (\eta^{\text{RK}} + \eta^{\text{KA}}) \quad (47)$$

$$\tilde{\Gamma}_{21}^1 = \frac{1}{\sqrt{2}} (1 - \eta^{\text{AR}})^{-1}, \quad (48)$$

so that  $\tilde{\Gamma}_{21}^1$  is simply a diffusion operator, as it is also manifest in Figure 7. The explicit space and time dependence



**Fig. 10.** The interaction vertex; compare Figure 7.

is determined as

$$\tilde{\Gamma}_{21}^1(x; x_1, x_2) = \frac{1}{\sqrt{2}} D_{tt_1}^{\eta=0}(\mathbf{x}, \mathbf{x}_1) \delta(\mathbf{x}_1 - \mathbf{x}_2) \delta(t_1 - t_2). \quad (49)$$

The vertex  $\Gamma_{12}^1$  has a more complicated structure due to the presence of a Keldysh Green function in the kernel of the corresponding integral equation. Its detailed diagrammatic representation is shown in Figure 10. The integral  $\eta^{\text{RK}}$  is given by

$$\eta^{\text{RK}} = \frac{1}{2\pi N_0 \tau} \int \frac{d^3p}{(2\pi)^3} G^R(\mathbf{p} + \mathbf{q}/2) \times G^K(\mathbf{p} + \mathbf{q}/2 - \mathbf{Q}; \mathbf{p} + \mathbf{q}_1 - \mathbf{q}/2 - \mathbf{Q}). \quad (50)$$

We replace  $G^K$  by  $G^R(-i/\tau) F G^A$ , with

$$F_{tt'}(\mathbf{x}) = \frac{1}{2} \int \frac{d\hat{p}}{4\pi} g_{tt'}^K(\mathbf{p}, \mathbf{x}) \quad (51)$$

and integrate over  $\mathbf{p}$  in equation (50) to get  $\eta_{tt'}^{\text{RK}}(\mathbf{x}) = F_{tt'}(\mathbf{x})$ . It can be easily shown that  $\eta^{\text{KA}}$  does not contribute to the current; the (relevant part of the) vertex is then found as

$$\Gamma_{12}^1(x; x_1, x_2) = \frac{1}{\sqrt{2}} \int d\eta D_{t_1-\eta/2, t-\eta/2}^\eta(\mathbf{x}_1, \mathbf{x}) \times F_{t, t-\eta}(\mathbf{x}) \delta(\mathbf{x}_1 - \mathbf{x}_2) \delta(t_2 - t_1 + \eta). \quad (52)$$

The last ingredient entering the expression of the interacting self-energy is the electron-electron interaction propagator  $V^{\text{R,A}}$ . At the level of the approximation we are working, it is sufficient to confine to the standard random phase approximation (RPA). The retarded RPA screened Coulomb interaction reads:

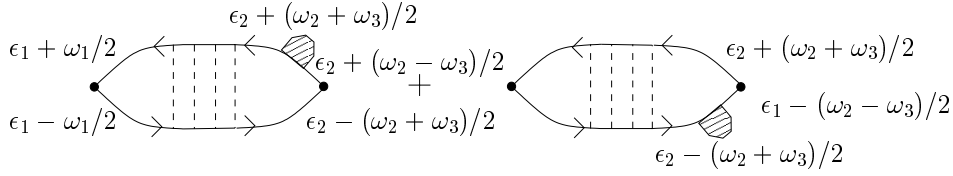
$$V^{\text{R}}(x, x') = V^0(x, x') - \int dx_1 dx_2 V^0(x, x_1) \Pi^{\text{R}}(x_1, x_2) V^{\text{R}}(x_2, x'), \quad (53)$$

where

$$V^0(x, x') = \delta(t - t') e^2 / |\mathbf{x} - \mathbf{x}'| \quad (54)$$

and  $\Pi^{\text{R}}(x_1, x_2)$  is the retarded component of the density correlation function. We write the density correlation function as the sum of a “static” and a “dynamic” part,  $\Pi^{\text{R}} = \Pi^{\text{s}} + \Pi^{\text{d}}$ . The “static” part is given by

$$\Pi^{\text{s}}(x_1, x_2) = 2N_0 \delta(\mathbf{x}_1 - \mathbf{x}_2) \delta(t_1 - t_2), \quad (55)$$



**Fig. 11.** The dynamical part of the density correlation function.

as it is seen directly from the quasi-classical expression for the electron density in equation (21). The “dynamic” part is determined from the ladder diagrams as shown in Figure 11. The expression is given by

$$\begin{aligned} \Pi_{tt'}^d(\mathbf{x}, \mathbf{x}') &= 2\pi i N_0 \tau \int \frac{d\epsilon_1}{2\pi} \dots \frac{d\omega_2}{2\pi} e^{-i\omega_1 t + i\omega_2 t'} \\ &\times D\left(\epsilon_1 + \frac{\omega_1}{2}, \epsilon_1 - \frac{\omega_1}{2}; \epsilon_3 + \frac{\omega_2}{2}, \epsilon_2 - \frac{\omega_2}{2}\right) \\ &\times \left[ F\left(\epsilon_3 + \frac{\omega_2}{2}, \epsilon_2 + \frac{\omega_2}{2}\right) - F\left(\epsilon_3 - \frac{\omega_2}{2}, \epsilon_2 - \frac{\omega_2}{2}\right) \right], \end{aligned} \quad (56)$$

where we suppressed the spatial indices.  $D(\dots)$  is the diffuson and  $F$  is the angular average over  $g^{\mathbf{k}}$  as defined in (51). For the frequencies in  $F$  we introduce the center-of-mass and relative coordinates,  $F(\epsilon_1, \epsilon_2) \rightarrow F_{(\epsilon_1 + \epsilon_2)/2}(\epsilon_1 - \epsilon_2)$ . In order to evaluate the frequency integrals we assume that the time dependent distribution function  $F_\epsilon(t)$  deviates from the equilibrium distribution function only at small energies  $\epsilon$ , *i.e.*  $F_\epsilon(t) \rightarrow \pm 1$  for  $\epsilon \rightarrow \pm\infty$ . We then find

$$\Pi_{t_1 t_2}^d(\mathbf{x}_1, \mathbf{x}_2) = 2N_0 \tau \frac{\partial}{\partial t_2} D_{t_1 t_2}^0(\mathbf{x}_1, \mathbf{x}_2). \quad (57)$$

Since here the diffuson enters with the relative time  $\eta = 0$ , the density correlation function does not depend on the external fields. Adding the static and dynamic parts together and making use of the differential equation for the diffuson, one finds

$$\Pi^R(x_1, x_2) = -2N_0 \tau D \partial_{\mathbf{x}_1}^2 D_{t_1 t_2}^0(\mathbf{x}_1, \mathbf{x}_2). \quad (58)$$

In the case of a uniform system this reduces to the standard expression

$$\Pi^R(\mathbf{q}, \omega) = 2N_0 \frac{Dq^2}{-i\omega + Dq^2} \quad (59)$$

from which the dynamically screened Coulomb interaction is determined as

$$V(\mathbf{q}, \omega) = \frac{V^0(q)}{1 + V^0(q)2N_0 Dq^2 / (-i\omega + Dq^2)} \quad (60)$$

$$\approx \frac{1}{2N_0} \frac{-i\omega + Dq^2}{Dq^2}. \quad (61)$$

The second line is valid, when  $|\omega| < V^0(q)2N_0 Dq^2$ . In three dimensions, where  $V^0(q) = 4\pi e^2/q^2$ , this condition reads  $|\omega| < D\kappa^2$ . Since the inverse screening length  $\kappa$  is in a metal typically of the order of the Fermi wavelength the

approximation is well justified. In lower dimensions a more careful analysis is sometimes necessary, see for example reference [27, 18].

If we now collect all the pieces of our analysis we may come back to the quantum correction to the current. For convenience we switch from the energy/momentum domain to the time/space domain. Remember that we have neglected the energy dependence of the Green function  $G^R$  and  $G^A$  in the calculation of the Hikami box. This is equivalent to approximate these Green functions as local in time. The resulting time dependencies are shown in Figures 9 and 10. The correction to the current  $\delta\mathbf{j} = \delta\mathbf{j}_a + \delta\mathbf{j}_b$  is finally found as

$$\delta\mathbf{j}_a(\mathbf{x}, t) = -D\nabla\delta\rho(\mathbf{x}, t) \quad (62)$$

$$\begin{aligned} \delta\mathbf{j}_b(\mathbf{x}, t) &= e2\pi DN_0 \tau^2 \int d\eta dx_1 dx_2 \\ &\times F_{t-\eta, t}(\mathbf{x}) D_{t-\eta/2, t_1-\eta/2}^\eta(\mathbf{x}, \mathbf{x}_1) F_{t_1, t_1-\eta}(\mathbf{x}_1) \\ &\times V_{t_1, t_2}^R(\mathbf{x}_1, \mathbf{x}_2) (-i\nabla_{\mathbf{x}}) D_{t_2, t-\eta}^0(\mathbf{x}_2, \mathbf{x}) + \text{c.c.} \end{aligned} \quad (63)$$

This current formula is one of the central results of our paper. The nice feature is that is valid for arbitrary form of the distribution function and diffuson propagator. This will allow us to examine the current in different experimental and geometrical setups as well as more general questions concerning its physical interpretations. Specific applications are discussed in Sections 4, 5. The interacting disordered electron problem is often formulated in terms of the field-theoretic non-linear sigma model and one may wonder how the presented diagrammatic approach is related to it. To this end we explicitly show in the Appendix A at the end of the paper that the same expression for the current can be obtained from the field theoretic approach of reference [28].

## 4 Gauge invariance

As a first application of the formula for the current we discuss in this section the issue of its physical interpretation and of gauge invariance. To begin with, we believe useful to make contact with our earlier work in references [18, 19].

In references [18, 19] the quantum correction to the current was derived working in a vector gauge,  $\mathbf{A} = -t\mathbf{E}$ ,  $\phi = 0$ , under the assumption that the electron distribution function had the equilibrium form,  $F(\epsilon, \mathbf{x}) = \tanh(\epsilon/2T)$ , or, equivalently, in the time domain  $F_{tt'}(\mathbf{x}) = -iT/\sinh[\pi T(t-t')]$ , and that we dealt with a uniform system with a homogeneous charge density. In this situation

one has immediately that  $\delta \mathbf{j}_a = 0$  and by Fourier transforming equation (63) with respect to the spatial variables, the current formula of [18,19] is reproduced.

In [18,19] we showed that in the limit of weak electric field our theory reproduces the well-known Altshuler-Aronov corrections to the conductivity. At larger fields non-linear contributions to the current arise as a consequence of the nonlocal character of the current formula.

By working in the vector gauge, the electric field enters the equation for the diffuson, equation (30), *via* the minimal substitution of the vector potential. Due to the quasi-classical nature of the equation governing the diffuson, one may interpret the non-linear conductivity in terms of phases. Let us recall the argument of reference [19]. The interaction correction to the conductivity is related to the propagation of a particle and a hole along closed paths. Pictorially one may think of this as one particle going around a closed path, starting for example at  $t = 0$  and arriving at  $t = \eta$ . This particle is also interacting with a background particle which is retracing backwards-in-time the same closed path. Since the point of interaction  $\mathbf{x}(t_1)$  can be anywhere along the path, the particles traverse the loop at different times. In the presence of a vector potential the accumulated phase difference of the two paths is  $\varphi_1 - \varphi_2 = e \int_{t_1-\eta}^{t_1} dt' \dot{\mathbf{x}}_1 \cdot \mathbf{A} - e \int_0^\eta dt' \dot{\mathbf{x}}_2 \cdot \mathbf{A}$ . This can be simplified using that  $\mathbf{x}_1(t) = \mathbf{x}_2(t)$  for  $0 < t < t_1$  and  $\mathbf{x}_1(t - \eta) = \mathbf{x}_2(t)$  for  $t_1 < t < \eta$  leading to  $\varphi_1 - \varphi_2 = e \int_{t_1-\eta}^0 dt' \dot{\mathbf{x}}_1 \cdot [\mathbf{A}(t') - \mathbf{A}(t' + \eta)]$ . For the particular case of a static electric field described by  $\mathbf{A} = -\mathbf{E}t$ , the above given phase difference becomes  $\varphi_1 - \varphi_2 = e\eta(\mathbf{x}_2 - \mathbf{x}_1) \cdot \mathbf{E}$ . This suggests that the interaction correction should be sensitive to a static electric field, leading to a non-linear conductivity.

One may object against this interpretation by observing that the vector potential can be gauged away in such a way that the static electric field is described by a static scalar potential  $E(\mathbf{r}) = -\nabla\phi(\mathbf{r})$ . A static scalar potential, again according to equation (30), does no longer affect the diffuson propagator, so that the argument of the phase difference along the two paths cannot be used. Of course this does not imply that the current formula is incorrect.

In fact we may demonstrate explicitly the gauge invariance of the current formula. First one notices that  $\delta \mathbf{j}_a = -D\nabla\rho$  is gauge invariant. For  $\delta \mathbf{j}_b$  an explicit check is necessary. Given the gauge transformation

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi \quad (64)$$

$$\phi \rightarrow \phi - \partial_t\chi \quad (65)$$

the diffuson and the distribution function transform according to

$$F_{tt'}(\mathbf{x}) \rightarrow F_{tt'}(\mathbf{x}) \exp\{-ie[\chi(\mathbf{x}, t) - \chi(\mathbf{x}, t')]\} \quad (66)$$

$$D_{tt'}^\eta(\mathbf{x}, \mathbf{x}') \rightarrow D_{tt'}^\eta(\mathbf{x}, \mathbf{x}') \quad (67)$$

$$\begin{aligned} & \times \exp\left\{-ie\left[\chi\left(\mathbf{x}, t + \frac{\eta}{2}\right) - \chi\left(\mathbf{x}, t - \frac{\eta}{2}\right)\right]\right\} \\ & \times \exp\left\{ie\left[\chi\left(\mathbf{x}', t' + \frac{\eta}{2}\right) - \chi\left(\mathbf{x}', t' - \frac{\eta}{2}\right)\right]\right\}. \end{aligned}$$

By applying the above transformation to equation (63), one easily verifies that the function  $\chi(\mathbf{x}, t)$  drops, so that the expression is manifestly gauge invariant. For the special example of a static electric field with  $\mathbf{A} = -\mathbf{E}t$  we choose  $\chi(\mathbf{x}, t) = t\mathbf{E} \cdot \mathbf{x}$ . After the gauge transformation the electric field appears in the distribution function  $\tanh(\epsilon/2T) \rightarrow \tanh[(\epsilon - \mu_{\mathbf{x}})/2T]$ ,  $\mu_{\mathbf{x}} = e\mathbf{E} \cdot \mathbf{x}$  but not in the diffuson.

We conclude that although the correction to the current as derived in this paper is gauge invariant, the interpretation of the non-linear effects depends on the actual choice of the potentials  $\mathbf{A}$  or  $\phi$ . With  $\mathbf{E} = -\partial_t\mathbf{A}$ ,  $\phi = 0$  we would interpret the non-linear conductivity as due to dephasing. With  $\mathbf{E} = -\nabla\phi$ ,  $\mathbf{A} = 0$  the reason of the non-linear conductivity is attributed to the different local chemical potential felt by the particle and hole.

## 5 Mesoscopic wire

In this section we use the current formula to analyze the non-linear electrical transport in a thin wire. We assume that the temperature is low enough so that the Drude conductivity is dominated by the impurity scattering and is therefore temperature independent. It is well known that in one dimension the electron-electron interaction leads to a  $1/\sqrt{T}$  correction to the conductivity. The interesting question to ask concerns what happens at larger voltages and which are the relevant length and energy scales in the problem.

For the calculation we need the diffuson propagator and the distribution function in the wire. At the boundary with the vacuum or an insulator the derivative of the diffuson normal to the boundary vanishes,  $(\mathbf{n} \cdot \nabla)D(\mathbf{x}, \mathbf{x}') = 0$ . In the case of an infinitely long wire with cross section  $S$  the solution of the diffusion equation reads

$$D(x, t) = \frac{1}{\tau} \frac{1}{S} \frac{1}{\sqrt{4\pi Dt}} \exp[-x^2/(4Dt)]. \quad (68)$$

In the above result we have averaged the diffuson over the cross section. In a wire of finite length we impose the open boundary conditions along the  $x$  axis

$$D_{tt'}(x, x')|_{x, x'=0, L} = 0, \quad (69)$$

corresponding to the fact that an electron arriving at the boundary escapes into the leads, where dissipation takes place. Therefore it no longer contributes to the phase coherent process of quantum interference. The diffuson in the finite system is related to the propagator for an infinite system according to

$$\begin{aligned} D_{tt'}(x, x') = & \sum_{n=-\infty}^{\infty} [D(x - x' + 2nL, t - t') \\ & - D(x + x' + 2nL, t - t')]. \end{aligned} \quad (70)$$



For the actual calculations it is convenient to consider also the product of the retarded interaction and the diffuson,

$$(V^R D)_{tt'}(\mathbf{x}, \mathbf{x}') = \int d\mathbf{x}_1 dt_1 V_{tt_1}^R(\mathbf{x}, \mathbf{x}_1) D_{t_1 t'}^0(\mathbf{x}_1, \mathbf{x}'). \quad (71)$$

For the case of long range interaction this product solves the equation

$$[-D\nabla_{\mathbf{x}}^2] (V^R D)_{tt'}(\mathbf{x}, \mathbf{x}') = \frac{1}{2N_0\tau} \delta(\mathbf{x} - \mathbf{x}') \delta(t - t'), \quad (72)$$

as it may be seen by comparing with equation (61). For a one dimensional wire with open boundary conditions we obtain then

$$(V^R D)_{tt'}(x, x') = \left[ \frac{(L - x')x}{L} - (x - x')\Theta(x - x') \right] \times \frac{1}{2DN_0\tau} \delta(t - t'). \quad (73)$$

Besides the boundary conditions for the diffuson and the interaction we need the boundary condition for the distribution function. We assume that the leads of the wire are in thermal equilibrium, so that the distribution function is given by

$$F(\epsilon, \mathbf{x})|_{\mathbf{x}=0,L} = \tanh\left(\frac{\epsilon \pm eV/2}{2T}\right). \quad (74)$$

The distribution function inside a mesoscopic wire has been investigated both experimentally [20] and theoretically [29,30]. In the theoretical analysis, in particular, a solution of the Boltzmann equation in the presence of disorder, electron-electron interaction and electron-phonon interaction has been given. Here we borrow the approximate solutions for the the distribution function found in references [29,30].

The form of the distribution function depends on the various relaxation mechanisms governing the collision integral. In the following we first consider a long wire,  $L \gg L_{\text{ph}}$ , and then subsequently reduce the length to  $L_{\text{ph}} \gg L \gg L_{\text{in}}$  and  $L_{\text{in}} \gg L \gg L_{\text{T}}$ . Here we indicate with  $L_{\text{ph}}$ ,  $L_{\text{in}}$ , and  $L_{\text{T}}$  the electron-phonon, the inelastic and the thermal scattering lengths.

## 5.1 Long wire

In the case of a long wire,  $L \gg L_{\text{ph}} \gg L_{\text{in}}$ , the electrons which traverse the wire scatter many times inelastically and exchange energy with the environment, for example with the phonons. As a result the distribution function acquires the equilibrium form with a local chemical potential and temperature. Our ansatz for the distribution function is

$$F(\epsilon, x) = \tanh\left(\frac{\epsilon + eV(L - 2x)/2L}{2T_e(x)}\right). \quad (75)$$

We assume that the temperature is constant in the bulk of the wire, and we also neglect the region near the leads where the electron temperature increases from the value in the leads to the one in the bulk. The electron temperature in the bulk may be estimated with standard energy balance arguments [16,13]. For a stationary temperature  $T_e$ , the Joule heating power  $P_{\text{in}} = \sigma \mathbf{E}^2$  equals the power which is transferred into the phonon system,  $P_{\text{out}}$ . For weak heating, one has  $P_{\text{out}} = c_V \Delta T / \tau_{\text{ph}}$ , where  $c_V$  is the electron specific heat. One obtains then that the difference of electron and phonon temperature may be estimated as

$$\Delta T \approx \frac{3}{\pi^2} D (eV/L)^2 \tau_{\text{ph}} / T. \quad (76)$$

For strong heating, on the other hand, the effective electron temperature is of the order of the voltage drop over a phonon length

$$T_e \sim eV L_{\text{ph}} / L. \quad (77)$$

We are now ready to evaluate the quantum correction to the current in the wire as

$$I = \frac{1}{L} S \int_0^L dx j^x(x), \quad (78)$$

where  $j^x$  is the component of the current parallel to the wire. Recall that we separated the correction to the current density into two contributions  $\delta \mathbf{j} = \delta \mathbf{j}_a + \delta \mathbf{j}_b$ . The first of the two terms does not contribute to the correction to the current since

$$\int_0^L dx \delta j_a \propto \int_0^L dx \frac{\partial}{\partial x} \delta g^K(x) \quad (79)$$

$$= \delta g^K(L) - \delta g^K(0) \quad (80)$$

and the boundary conditions impose that  $\delta g^K$  vanishes on the leads. The correction to the current then reads

$$I = \frac{2\pi e\tau}{L} \int_0^L dx dx_1 \int d\eta \text{Re} \left\{ F_{t-\eta,t}(x) F_{t-\eta,t-2\eta}(x_1) \times D_{t-\eta/2,t-3\eta/2}^\eta(x, x_1) (-i) \left[ \Theta(x_1 - x) - \frac{x_1}{L} \right] \right\}. \quad (81)$$

It is useful to introduce the center-of-mass and relative coordinate  $R = (x + x_1)/2$ ,  $r = x_1 - x$ . The last term in equation (81) above becomes then  $\Theta(r) - R/L - r/2L$ . The term  $R/L$  vanishes upon integration due to the antisymmetry of the  $r$ -integral. The assumption that the thermal length is much shorter than the system size allows furthermore to approximate  $\Theta(r) - r/2L$  by  $\Theta(r)$ , to neglect the boundary effect on the diffuson, *i.e.*  $D_{t-\eta/2,t-3\eta/2}^\eta(x, x_1) \approx D(r, \eta)$ , and to extend the  $r$ -integration to infinity. By inserting the distribution function we arrive at

$$\delta I = -2\pi e\tau \int_0^\infty dr \int_0^\infty d\eta \left( \frac{T_e}{\sinh(\pi T_e \eta)} \right)^2 \times D(r, \eta) \sin(eV r \eta / L). \quad (82)$$

This is equivalent to what is obtained in reference [18]. In the limit of low voltage the current becomes

$$\delta I(V) \approx \frac{e^2}{\pi^2} \frac{\sqrt{D/T_e}}{L} V \left( -4.92 + 0.21 \frac{D(eV/L)^2}{T_e^3} + \dots \right), \quad (83)$$

where 4.92 and 0.21 are approximate numerical factors. In order to obtain the full current, one has to add the contribution of the Drude leading term, *i.e.*,  $I = 2e^2 DN_0 SV/L + \delta I$ . The low voltage expansion applies when the voltage drop over a thermal length is smaller than the temperature,  $eVL_T/L < T_e$ . Since we have assumed that  $L_{ph} \gg L_T$ , the electron temperature as a function of voltage rises so fast that the condition always holds. Finally we compare the heating and non-heating contribution to the non-linear conductivity at low voltage. By taking the linear conductivity and the increase in temperature due to low voltages from equation (76), we find the heating contribution to the cubic term in the current voltage characteristics in the form

$$\delta I_{\text{heating}} \approx 4.92 \frac{e^2}{\pi^2} \frac{L_T}{L} V \frac{3}{2\pi^2} \frac{D(eV/L)^2}{T^2(1/\tau_{ph})}. \quad (84)$$

This has to be compared with the corresponding non-heating cubic contribution

$$\delta I_{\text{non-heating}} \approx 0.21 \frac{e^2}{\pi^2} \frac{L_T}{L} V \frac{D(eV/L)^2}{T^3}. \quad (85)$$

One observes that the heating contribution is by a factor of the order of  $T\tau_{ph}$  larger than the non-heating contribution.

## 5.2 Intermediate length

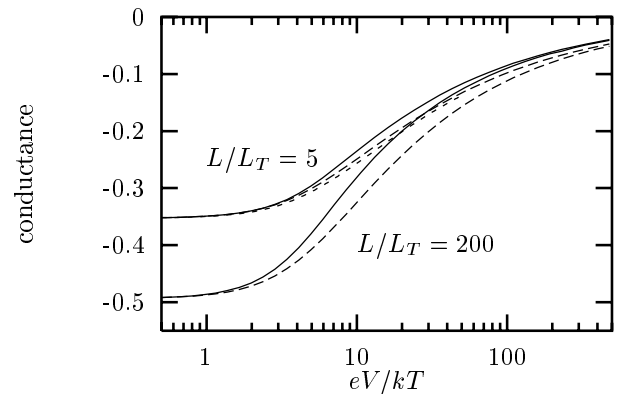
Now we consider a wire of intermediate length where  $L_{ph} \gg L \gg L_{in}$ . One still expects to be near local equilibrium although the main mechanism which carries the energy out of the wire is not due to the phonons, but to the heat flow out of the wire. Under these conditions, a temperature profile over the wire develops. The local temperature satisfies the equation [31]

$$\frac{d^2}{dx^2} T_e^2(x) = -\frac{6}{\pi^2} \frac{(eV)^2}{L^2} \quad (86)$$

which is solved by

$$T_e^2(x) = T^2 + \frac{3}{\pi^2} \left( \frac{eV}{L} \right)^2 x(L-x). \quad (87)$$

The correction to the current at a point  $x$  probes the wire in a region  $\sim L_T$  around  $x$ . As a consequence, non-linearities in the  $I - V$  characteristics arise because (1) the temperature depends on the voltage (“heating”), and of (2) the non-local character of the current formula (“non-heating”). It turns out that the first effect dominates, whereas the non-heating contribution to the non-linear conductivity is only a small perturbation. For illustration, notice that the heating becomes strong at



**Fig. 12.** Interaction correction to the conductance  $I/V$  for a mesoscopic wire as a function of voltage.  $I/V$  is plotted in units of  $(e^2/\hbar)L_T/L$ . The full line corresponds to the non-equilibrium distribution function (89). The long dashed line corresponds to the local equilibrium distribution function (75) with the  $x$ -dependent temperature. The short dashed line ( $L/L_T = 5$ ) is the non-linear conductivity due to the heating contribution only, equation (88).

$eV \sim kT$ . Non-heating non-linearities, on the other hand, arise on the scale  $eVL_T/L \sim kT$ . Since we assume that  $L \gg L_T$ , heating is indeed dominant. This is also demonstrated in Figure 12 where we plot the conductance  $\delta G = \delta I/V$  as a function of voltage, while varying the system size,  $L/L_T$ , and the distribution function  $F_{tt'}(x)$ . For large  $L/L_T$  the linear conductance approaches  $\delta I/V \approx -(4.92e^2/\pi^2)(L_T/L)V$ . For the smaller system size  $L/L_T = 5$ , the linear conductance is suppressed, due to the chosen boundary conditions. The long dashed lines and the full lines correspond to the local equilibrium distribution (75), and to the non-equilibrium distribution discussed below, respectively. The short dashed line ( $L/L_T = 5$ ), instead, is obtained by taking into account only the heating contribution, *i.e.* by calculating from equation (81) the *linear* conductivity and averaging over the  $x$ -dependent temperature,

$$\delta I_{\text{heating}} = \frac{1}{L} \int_0^L dx \delta \sigma(T(x))(V/L). \quad (88)$$

One observes in Figure 12 that the “heating” contribution reproduces with a good accuracy the much more complicated full calculation. For the longer system with  $L/L_T = 200$ , we do not plot the “heating” curve, because in this case it is practically indistinguishable from the long dashed one. A slightly larger non-linear conductivity is found in the non-equilibrium situation as it will be discussed in the next section.

## 5.3 Short wire

In a very short wire the inelastic length may exceed the system size  $L$ . When one neglects the inelastic scattering, the distribution function inside the wire becomes a linear

superposition of the distribution functions in the leads and reads [21]

$$F(\epsilon, x) = [(L - x)F(\epsilon, 0) + xF(\epsilon, L)] / L. \quad (89)$$

In the limit  $L \gg L_T$  the analytic calculation of the current proceeds as in the case of Section 5.1. In analogy to (82) we arrive at

$$\delta I = -2\pi e\tau \int_0^\infty dr \int_0^\infty d\eta \left( \frac{T}{\sinh(\pi T\eta)} \right)^2 \times D(r, \eta) \sin(eV\eta)r/L. \quad (90)$$

The numerical results for the current-voltage characteristics in the presence of such a distribution function are shown in Figure 12. Notice that we integrated numerically equation (81), as it is appropriate when  $L_T/L$  is not very large. The linear conductance is – of course – the same as in the local equilibrium situation, whereas the non-linear effects are slightly larger.

#### 5.4 The Spin-triplet channel

Up to now we have neglected all the spin effects. As demonstrated in Appendix B, the current formula can easily be generalized in order to include also the so called spin triplet channels. The general equation for the correction to the current is given in the appendix. With the two-step distribution function (89) and for  $L \gg L_T$  the equation for the current reads

$$\delta \mathbf{j}_b = -4\pi eDN_0\tau^2 \sum_{i=0}^3 \int_{-\infty}^\infty dr \int_0^\infty d\eta \int_0^\eta dt_1 \times \left( \frac{T}{\sinh(\pi T\eta)} \right)^2 \sin(eV\eta) \frac{r}{L} \times D(r, t_1) \frac{\partial}{\partial r} (\tilde{\gamma}^i D)(r, \eta - t_1), \quad (91)$$

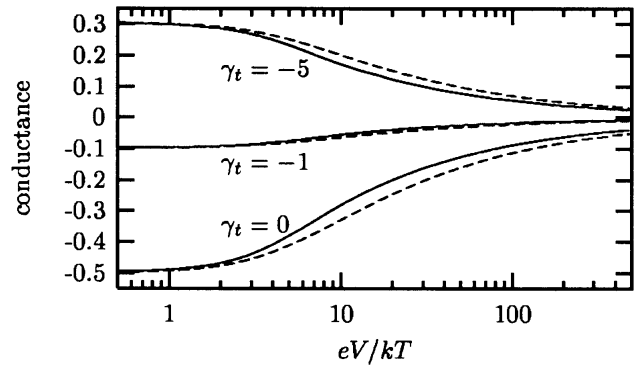
with

$$(\tilde{\gamma}^i D)(r, t) = \frac{\gamma^i}{\tau} \sqrt{\frac{1 - 2\gamma_i}{4\pi Dt}} \exp[-r^2(1 - 2\gamma_i)/4Dt], \quad (92)$$

and  $\gamma^i$  are the interaction amplitudes in the spin singlet ( $\gamma^0 = 1/2$ ) and spin triplet ( $\gamma^{1,2,3} = \gamma^t$ ) channels. In the low voltage limit this expression reproduces the standard result for the linear conductivity,

$$\delta\sigma = -4.92 \frac{e^2}{\pi^2} \frac{L_T}{L} \left[ 1 - 3(\sqrt{1 - 2\gamma_t} - 1 + \gamma_t)/\gamma_t \right]. \quad (93)$$

Figure 13 depicts the current voltage characteristics for different strengths of the triplet scattering amplitude. Again we compare the full result (91) with the simple heating contribution, that is the average of the linear conductivity over the temperature profile given in (87). For strong scattering in the triplet channel the quantum correction changes sign. As a function of voltage, the quantum corrections are suppressed. For the case of a distribution function out of equilibrium, the non-heating non-linear contributions are stronger than the pure heating effects.



**Fig. 13.** Correction to the conductance as a function of voltage. Notice that for a strong negative scattering amplitude in the spin triplet channel,  $\gamma_t$ , the quantum correction changes sign. The dashed line corresponds to the hot electron regime (heating), while the full line to the non-equilibrium situation.

## 6 Conclusions

We have calculated the interaction correction to the electrical current for a disordered metal out of equilibrium. The current is expressed as a function of the (ensemble averaged) distribution function and the diffusion propagators. In order to obtain this result we have extended the diagrammatic approach of references [18, 19]. We have first demonstrated explicitly the gauge invariance of our current formula for the current density. In particular, we have discussed how the physical interpretation of the non-linear contribution may be described differently depending on the gauge choice made. In the scalar gauge one may think as the particle-hole pair lying on a different chemical potential. In the vector gauge, the chemical potential of the particle and hole are equal. In the latter case an argument based on phase differences leads to the conclusion that the electric field affects the correction to the conductivity.

We have successively discussed in some detail the correction for a mesoscopic wire. We have assumed that the wire is attached to “ideal leads” (infinite conductance), which are described by means of boundary conditions for the distribution function and the diffusion propagator. We have distinguished three different regimes. First we concentrated on a wire near local equilibrium, with a constant electron temperature. Besides heating, there is a contribution to the non-linear conductivity due to the nonlocal nature of the current response.

For short wires, where the electron-phonon length is longer than the system size, we found that the conductance scales with voltage over temperature. The quantitative shape of the “scaling curve” is not universal. We have found two different curves in the hot electron regime, where the inelastic scattering length is shorter than the system size, and in non-equilibrium. The scaling of the voltage with the temperature has been recently observed in a nanobridge [32]. Notice however that for a quantitative explanation of that experiment it might be important

to take into account scattering of the electrons at the interface between the leads and the bridge [32] or charging effects [33].

We come now back to the question concerning the origin of the saturation of the dephasing time and resistance in gold wires which originally motivated our investigations. We have shown that the non-linear conductivity for a wire near local equilibrium is dominated by heating, even in the case of a space dependent temperature. In the case of full non-equilibrium (double step distribution function) one finds only slightly enhanced non-linear effects. We have to conclude that a saturation of the Altshuler-Aronov correction due to non-equilibrium electric fields is most probably due to heating.

We would like finally to emphasize that in this paper we have concentrated on the quantum correction to the (ensemble averaged) current, and we have left aside a derivation of the charge density and the kinetic equation including the quantum corrections. This latter task implies the evaluation of the quantum correction to the distribution function. The evaluation of the distribution function and charge density are directly connected as it is clear when one expresses the distribution function in terms of the quasi-classical Green functions as  $g^K = g^R F - F g^A$ , and recalls the relation of  $g^K$  with the charge density. The inclusion of quantum corrections into the kinetic equation for disordered electrons have been considered for the weak localization case without electron interaction in references [35, 36], and in the presence of interactions in reference [37]. In the latter work, the non-linear electric field effects have not been included. We notice that the quantum correction to the charge density out of equilibrium can be derived following the procedure described in this paper for the current density in Section 3. This task is however more involved for the following reason. Within the leading order in the gradient expansion, one finds from  $\delta\Sigma^{\text{imp},K}$ , in analogy to equation (38), the identity  $\delta\rho_a = \delta\rho$ . The diagrams in Figure 9, responsible for the contribution to the current denominated  $\delta j_b$ , give in the case of the charge density  $\delta\rho_b = 0$ . This happens because, the evaluation of the Hikami box with the density vertex is zero in the leading order of the gradient expansion. In order to consider the next-to-leading terms, one has to take into account higher powers in the inverse of  $\epsilon_F\tau$  and  $ql$ . This requires the evaluation of also the diagrams with only one vertex,  $I$  or  $\tilde{I}$ , renormalized by the diffusion pole. Therefore the full expression of the electron self-energy is considerably more complicated and such a calculation will be more lengthy than the one we presented here for the current density. This task, although worth to be done, is beyond the scope of the present paper.

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## Appendix A: Field theoretic approach: the non-linear sigma model

The field theoretic formulation of the interacting, disordered electron system was pioneered by Finkelstein in the 80's [34]. Recently, this formulation has been extended to the non-equilibrium case by means of the Keldysh technique by Kamenev and Andreev [28] and Chamon *et al.* [38] for the case of normal-conducting metals and by Feigel'man, Larkin and Skvortsov [39] for superconductors. Gutman and Gefen [23] have also used the field theoretic description to calculate the current and zero frequency shot noise.

Given the already extensive literature available on the subject, we believe that, rather than repeating again the derivation of the non-linear sigma model, it is perhaps more useful, instead, to show how to obtain our current formula (63) within the non-linear sigma model. In this appendix we will discuss the spinless version of the model, following reference [28], postponing the spin effects to the following appendix. In the absence of the electron interaction, the action of the non-linear sigma model is given by

$$iS_0 = -\frac{\pi N_0}{4} [D\text{Tr}(\partial_{\mathbf{x}}Q)^2 + 4i\text{Tr}\epsilon Q], \quad (94)$$

where the so-called long derivative  $\partial_{\mathbf{x}}$  is defined by

$$\partial_{\mathbf{x}}Q = \nabla Q + ie[\mathbf{A}, Q]. \quad (95)$$

The field  $Q \equiv Q_{tt'}^{ij}(\mathbf{x})$  must satisfy the constraints  $Q^2 = 1$  and  $\text{Tr}Q = 0$ . The electron interaction is described by the following term in the action

$$iS_1 = -i\pi N_0 \text{Tr}\Phi_{\alpha}\gamma^{\alpha}Q, \quad (96)$$

with  $\gamma^1 = \sigma_0$  and  $\gamma^2 = \sigma_x$ . The fluctuations of the field  $\Phi$  are related to the statically screened Coulomb interaction and given by

$$-i\langle\Phi_i(\mathbf{x}, t)\Phi_j(\mathbf{x}', t')\rangle = \frac{1}{2}V\sigma_x^{ij}\delta(\mathbf{x} - \mathbf{x}')\delta(t - t'). \quad (97)$$

In the case of long range Coulomb forces one finds  $V = 1/N_0$ . Since  $\Phi$  couples only linearly to  $Q$  it can be integrated out:

$$\langle e^{-i\pi N_0 \text{Tr}\Phi_{\alpha}\gamma^{\alpha}Q} \rangle_{\Phi} = \exp\left[-\frac{iV(\pi N_0)^2}{2} \int d\mathbf{x}dt \text{Tr}(\gamma^1 Q_{tt}(\mathbf{x}))\text{Tr}(\gamma^2 Q_{tt}(\mathbf{x}))\right]. \quad (98)$$

Here the trace refers to the Keldysh space. The appearance of the product of terms containing  $\gamma^1$  and  $\gamma^2$  stems from the  $\sigma_x$  structure of the interaction matrix in Keldysh space. In order to make contact with the diagrammatic approach, we express the Green functions in terms of the  $Q$ -fields. First we observe that [28]

$$\hat{G}_{tt'}(\mathbf{x}, \mathbf{x}') = \left\langle \left[ G_0^{-1} + \frac{i}{2\tau}Q + \Phi_{\alpha}\gamma^{\alpha} \right]^{-1} \right\rangle_{Q, \Phi}, \quad (99)$$

where the brackets  $\langle \dots \rangle_{Q, \Phi}$  indicate that one has to average over the fields  $Q$  and  $\Phi$ . For the sake of simplicity, we drop the subscripts  $Q, \Phi$  in the following. By using the condition that the fields  $Q$  and  $\Phi$  are only slowly varying in space and time, one finds a relation between the  $\xi$ -integrated Green function and the  $Q$ -matrix. In particular, upon taking the  $s$ -wave component of both, one finds

$$\int \frac{d\hat{p}}{4\pi} \hat{g}_{tt'}(\hat{\mathbf{p}}, \mathbf{x}) = \langle \hat{Q}_{tt'}(\mathbf{x}) \rangle. \quad (100)$$

In a similar way the  $p$ -wave part of the  $\xi$ -integrated Green function is related to  $Q$  by

$$v_F \int \frac{d\hat{p}}{4\pi} \hat{p} \hat{g}_{\epsilon\epsilon'}(\hat{\mathbf{p}}, \mathbf{x}) = \frac{1}{2} D \langle \partial_{\mathbf{x}} Q Q - Q \partial_{\mathbf{x}} Q \rangle, \quad (101)$$

so that the current density reads

$$\mathbf{j} = \frac{1}{2} e\pi D N_0 \langle (\partial_{\mathbf{x}} Q Q - Q \partial_{\mathbf{x}} Q)^2 \rangle. \quad (102)$$

At a first glance, this might differ from what is found following reference [28], where the current density is written as

$$\mathbf{j} = \frac{1}{2} e\pi D N_0 \langle \text{Tr} \gamma^2 (\partial_{\mathbf{x}} Q Q - Q \partial_{\mathbf{x}} Q) \rangle. \quad (103)$$

On the other hand, by comparing with the expression (101) for the Green function, it is seen that equation (103) sums the Keldysh component and the (21)-component of the Green function, whereas (102) takes only the Keldysh component. Since the (21)-component is zero the two expressions are equivalent.

## A.1 Propagators

The saddle point approximation for the  $Q$ -field

$$Q^{\text{sp}} = \begin{pmatrix} 1 & 2F \\ 0 & -1 \end{pmatrix} \quad (104)$$

reproduces the Drude-Boltzmann theory. The quantum corrections are found when considering the fluctuations about the saddle point. We parameterize  $Q$  according to

$$Q = u e^{-W/2} \sigma_z e^{W/2} u \quad (105)$$

where  $u$  characterizes the saddle-point distribution function

$$u = \begin{pmatrix} 1 & F \\ 0 & -1 \end{pmatrix}, \quad Q^{\text{sp}} = u \sigma_z u, \quad (106)$$

and  $W$  parameterizes the fluctuations,

$$W = \begin{pmatrix} 0 & w \\ \bar{w} & 0 \end{pmatrix}. \quad (107)$$

By expanding in powers of the  $W$ -field, one finds for the non-interacting action,  $S_0$ , up to the quadratic order

$$iS_0^{(2)} = -\frac{\pi N_0}{2} \left\{ \int d\mathbf{x} dt_1 dt_2 w_{t_1 t_2} (\partial_{t_1} + \partial_{t_2} + D \partial_{\mathbf{x}}^2) \bar{w}_{t_2 t_1} + \int d\mathbf{x} dt_1 \dots dt'_2 \bar{w}_{t_1 t_2} (\partial_{\mathbf{x}} F_{t_2 t'_2}) \bar{w}_{t'_2 t'_1} (\partial_{\mathbf{x}} F_{t'_1 t_1}) \right\}. \quad (108)$$

The long derivative  $\partial_{\mathbf{x}} \bar{w} = \nabla \bar{w} + ie[\mathbf{A}, \bar{w}]$  can here also be written as  $\partial_{\mathbf{x}} = \nabla - ie\mathbf{A}_{t_1}(\mathbf{x}) + ie\mathbf{A}_{t_2}(\mathbf{x})$ . The  $\langle \omega \bar{\omega} \rangle$  correlations solve the differential equation

$$\left( -\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + D \partial_{\mathbf{x}}^2 \right) \langle w_{t_2 t_1}(\mathbf{x}) \bar{w}_{t_3 t_4}(\mathbf{x}') \rangle = \frac{2}{\pi N_0} \delta(\mathbf{x} - \mathbf{x}') \delta(t_1 - t_3) \delta(t_2 - t_4). \quad (109)$$

After introducing the relative times  $\eta = t_2 - t_1$ ,  $\eta' = t_4 - t_3$  and the center-of-mass times,  $t = (t_1 + t_2)/2$ ,  $t' = (t_3 + t_4)/2$ , we identify this correlator with the diffuson,

$$\langle w_{t_2 t_1}(\mathbf{x}) \bar{w}_{t_3 t_4}(\mathbf{x}') \rangle = -\frac{2\tau}{\pi N_0} D_{tt'}^{\eta}(\mathbf{x}, \mathbf{x}') \delta(\eta - \eta'), \quad (110)$$

as it may be seen by comparing with equation (30). Notice that the relative time  $\eta$  is conserved during the propagation so that  $\eta = \eta'$ . The  $\langle \bar{w} w \rangle$  correlator is the advanced counterpart of the diffuson.  $\langle \bar{w} \bar{w} \rangle = 0$  since there is no term proportional to  $w$  in the action. Finally the  $\langle w w \rangle$  correlator is given by

$$\langle w_{t_2 t_1} w_{t_3 t_4} \rangle = -\pi N_0 \int dt'_1 \dots dt'_4 \langle w_{t_2 t_1} \bar{w}_{t'_1 t'_2} \rangle \langle \bar{w}_{t'_4 t'_3} w_{t_3 t_4} \rangle \times \partial_{\mathbf{x}} F_{t'_2 t'_4} \partial_{\mathbf{x}} F_{t'_3 t'_1}. \quad (111)$$

This correlator is zero in equilibrium, when  $\partial_{\mathbf{x}} F = 0$ . Electron interactions modify these propagators. By expanding in (98) each  $Q$ -field to first order in  $W$ , the interacting part of the action becomes

$$= -i(\pi N_0)^2 \frac{1}{2} V \int dt (\bar{w} F - F \bar{w})_{tt} (\bar{w} - w - F \bar{w} F)_{tt}, \quad (112)$$

which leads to non-trivial modifications of all the  $\langle w w \rangle$  correlators. Only  $\langle \bar{w} \bar{w} \rangle$  is not modified by the interaction and remains zero. We consider now  $\langle w \bar{w} \rangle$ , which in the absence of interaction is just the diffuson. We find

$$\left( -\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + D \partial_{\mathbf{x}}^2 \right) \langle w_{t_2 t_1} \bar{w}_{t_3 t_4} \rangle_{\Phi} + i\pi N_0 V F_{t_2 t_1} (\langle w_{t_2 t_2} \bar{w}_{t_3 t_4} \rangle_{\Phi} - \langle w_{t_1 t_1} \bar{w}_{t_3 t_4} \rangle_{\Phi}) = \frac{2}{\pi N_0} \delta(\mathbf{x} - \mathbf{x}') \delta(t_1 - t_3) \delta(t_2 - t_4). \quad (113)$$

To understand the meaning of the above equation, let us consider first the limit  $t_2 \rightarrow t_1$ . The interaction dependent term does not drop from (113) since the distribution function is singular in this limit,  $F_{t_2 t_1} \approx -i/\pi(t_2 - t_1)$ .

Upon using this relation we arrive at the following simple diffusion equation

$$\left[ -(1 - N_0 V) \frac{\partial}{\partial t} + D \partial_{\mathbf{x}}^2 \right] \langle w_{tt} \bar{w}_{t_3 t_4} \rangle_{\Phi} = \frac{2}{\pi N_0} \delta(\mathbf{x} - \mathbf{x}') \delta(t - t_3) \delta(t - t_4). \quad (114)$$

To make contact with the diagrammatic calculation, we identify this correlation function with the product of the retarded interaction and diffusion propagators which have been defined in equation (71)

$$-\frac{\pi}{2} \frac{N_0 V}{2} \langle w_{tt} \bar{w}_{t_3 t_4} \rangle_{\Phi} = (V^R D)_{tt_3} \delta(t_3 - t_4). \quad (115)$$

The propagator with four different time arguments is finally given by

$$\langle w_{t_2 t_1} \bar{w}_{t_3 t_4} \rangle_{\Phi} = \langle w_{t_2 t_1} \bar{w}_{t_3 t_4} \rangle - i\pi N_0 V \int d\mathbf{x}' dt'_3 dt'_4 \times \left[ \langle w_{t_2 t_1} \bar{w}_{t'_3 t'_4} \rangle F_{t'_4 t'_3} \left( \langle w_{t'_4 t'_4} \bar{w}_{t_3 t_4} \rangle_{\Phi} - \langle w_{t'_3 t'_3} \bar{w}_{t_3 t_4} \rangle_{\Phi} \right) \right]. \quad (116)$$

## A.2 Correction to the current

Due to the fluctuations of  $Q$  there are corrections to the charge and current density. Here we calculate these corrections by taking into account the Gaussian fluctuations. In the derivation, we parallel the lines of the diagrammatic approach. We begin by separating the correction to the current in two contributions  $\delta\mathbf{j} = \delta\mathbf{j}_a + \delta\mathbf{j}_b$ , where  $\delta\mathbf{j}_a$  is related to the gradient of the charge density. We then proceed by expressing  $\delta\mathbf{j}_b$  in terms of the fields  $\bar{w}$  and  $w$ . In the third step we will then explicitly include the interaction and obtain the current formula. By writing  $Q = Q^{\text{sp}} + \delta Q$ , the correction to the current reads

$$\delta\mathbf{j}(\mathbf{x}, t) = -\frac{e\pi D N_0}{2} \langle \text{Tr} \gamma^2 \{ Q^{\text{sp}} \partial_{\mathbf{x}} \delta Q + \delta Q \partial_{\mathbf{x}} Q^{\text{sp}} + \delta Q \partial_{\mathbf{x}} \delta Q - \dots \} \rangle. \quad (117)$$

The dots correspond to the terms which appear due to  $(\partial_{\mathbf{x}} Q)Q$ . The term  $j_a$  is proportional to the gradient of the charge density

$$\delta\mathbf{j}_a(\mathbf{x}, t) = -e\pi D N_0 \nabla \langle \delta Q_{tt}^{12}(\mathbf{x}) \rangle \quad (118)$$

$$= -D \nabla \delta \rho(\mathbf{x}, t) \quad (119)$$

and we do not calculate it explicitly here. The second term,  $\delta\mathbf{j}_b$ , contains all other contributions. By collecting the various summands, one gets

$$\delta\mathbf{j}_b = 2F \partial_{\mathbf{x}} \langle \delta Q^{22} \rangle + \langle \delta Q^{11} \rangle \partial_{\mathbf{x}} (2F) + \langle \delta Q^{11} \partial_{\mathbf{x}} \delta Q^{12} \rangle + \langle \delta Q^{12} \partial_{\mathbf{x}} \delta Q^{22} \rangle - \dots \quad (120)$$

Now we expand  $Q$  in powers of  $W$  and we arrive at

$$\delta\mathbf{j}_b = e\pi D N_0 [F \langle (\partial_{\mathbf{x}} \bar{w}) w \rangle_{\Phi} + \langle w (\partial_{\mathbf{x}} \bar{w}) \rangle_{\Phi} F] \quad (121)$$

$$= 2\pi e D N_0 \text{Re} \langle (w \partial_{\mathbf{x}} \bar{w}) \rangle_{\Phi} F. \quad (122)$$

In the absence of interactions  $\delta\mathbf{j}_b$  vanishes, so that taking the interactions into account is essential. Inserting  $\langle w \bar{w} \rangle_{\Phi}$  from equation (116) we find

$$\delta\mathbf{j}_b = 4\pi e D N_0 \tau^2 \int d\eta dx_1 dx_2 \text{Re} \{ F_{t-\eta, t}(\mathbf{x}) \times D_{t-\eta/2, t_1-\eta/2}^{\eta}(\mathbf{x}, \mathbf{x}_1) F_{t_1, t_1-\eta}(\mathbf{x}_1) \times V_{t_1 t_2}^R(\mathbf{x}_1, \mathbf{x}_2) (-i \nabla_{\mathbf{x}}) D_{t_2, t-\eta}^0(\mathbf{x}_2, \mathbf{x}) \}, \quad (123)$$

which is identical to what we obtained in equation (62).

Notice that the long derivative in the current formula (122) reduces to the gradient in the final result. This happens since the two time indices in the relevant  $\bar{w}$  field are always equal,  $\langle w_{tt} \bar{w}_{t_3 t_4} \rangle_{\Phi} \propto \delta(t_3 - t_4)$ .

## Appendix B: Spin-Triplet channel

Until now we have neglected the spin effects. These arise from the fact that the diffusion is described by the particle-hole propagator, which may occur in four spin states depending on the relative spin of the particle and hole. In the case of Coulomb long range forces, the most singular contribution comes from the singlet channel that we have discussed throughout the paper. In addition to the singlet channel, there are three triplet channels which also contribute to the quantum corrections to the current. In this appendix we extend our work (see reference [19]) to a non-equilibrium situation.

In order to take into account the spin-dependent interactions, we follow references [34,38] and start from an interaction which is local in space and time. The fermionic action is of the type

$$iS_{\text{ee}} = -\frac{i}{2N_0} \text{Tr} \{ \Gamma \bar{\Psi}_s \bar{\Psi}_{s'} \Psi_s \bar{\Psi}_s - \Gamma_2 \bar{\Psi}_s \bar{\Psi}_{s'} \Psi_s \Psi_{s'} \}, \quad (124)$$

where  $\Gamma$  and  $\Gamma_2$  are dimensionless static scattering amplitudes, and  $\Psi_s$  is an operator for a Fermion with spin  $s$ . The trace includes integration over space and time contour, as well as summation over spin. The interaction can be written in terms of the charge and spin densities,

$$iS_{\text{ee}} = -\frac{i}{2N_0} \text{Tr} \left\{ \left( \Gamma - \frac{\Gamma_2}{2} \right) \rho \rho - \frac{\Gamma_2}{2} \mathbf{s} \cdot \mathbf{s} \right\}, \quad (125)$$

with  $\rho = \sum_s \bar{\Psi}_s \Psi_s$  and  $s^i = \sum_{ss'} \bar{\Psi}_s \sigma_{ss'}^i \Psi_{s'}$ . At this point it becomes convenient to introduce the interaction amplitudes in the charge (singlet) channel,  $\Gamma^s = \Gamma - \Gamma_2/2$  and in the spin (triplet) channel,  $\Gamma^t = -\Gamma_2/2$ . Then we decouple the interaction with the fields  $(\bar{\Phi}, \mathbf{B})$  for the charge and spin. By going through the steps of the derivation of

the  $\sigma$ -model one finds the following modifications of the action

$$iS_0 \rightarrow -\frac{\pi N_0}{4} [D \text{Tr} \partial_{\mathbf{x}} Q_{ss'} \partial_{\mathbf{x}} Q_{s's} + 4i \text{Tr} Z \epsilon Q_{ss}] \quad (126)$$

$$iS_1 \rightarrow -i\pi N_0 \text{Tr} [(\Phi_{\alpha} \delta_{ss'} + \mathbf{B}_{\alpha} \cdot \sigma_{ss'}) \gamma^{\alpha} Q_{s's}], \quad (127)$$

with

$$-i\langle \phi_{\alpha} \phi_{\beta} \rangle = \frac{1}{2} \frac{\Gamma_s}{N_0} \sigma_{\alpha\beta}^x \quad (128)$$

$$-i\langle B_{\alpha}^i B_{\beta}^j \rangle = \frac{1}{2} \frac{\Gamma_t}{N_0} \sigma_{\alpha\beta}^x \delta_{ij}. \quad (129)$$

In (126) we introduced the factor  $Z$  which arises in the renormalization of the  $\sigma$ -model [34]. One observes that  $Z$  can be absorbed in a redefinition of the interaction amplitudes, the quasi-particle diffusion constant, and the quasi-particle density of states according to

$$\Gamma^{s,t} \rightarrow \gamma^{s,t} = \Gamma^{s,t} / Z \quad (130)$$

$$D \rightarrow D_{\text{qp}} = D / Z \quad (131)$$

$$N_0 \rightarrow N_{\text{qp}} = N_0 Z. \quad (132)$$

We are now ready to consider the fluctuations. In analogy to what has been done in the spinless case, we introduce the charge and spin components of the fields  $w$  and  $\bar{w}$ ,

$$w_{ss'} = \frac{1}{\sqrt{2}} \sum_{i=0}^3 w^i \sigma_{ss'}^i \quad (133)$$

$$w^i = \frac{1}{\sqrt{2}} \sum_{ss'} w_{ss'} \sigma_{ss'}^i \quad (134)$$

where  $\sigma_{ss'}^0 = \delta_{ss'}$  and  $\sigma_{ss'}^i$  for  $i = 1, 2, 3$  are the usual Pauli matrices. In the quadratic fluctuations of the non-interacting action,  $S_0$ , the spin and charge terms decouple, since the structure is of the type  $\sum_i \bar{w}_{ss'}(\dots) w_{s's} = \sum_i \bar{w}^i(\dots) w^i$ . The coupling term  $iS_1$  is to first order in  $W$  given by

$$iS_1 = -i\pi N_0 \sqrt{2} \sum_i \text{Tr} \left\{ B_{\alpha}^i \gamma^{\alpha} u \begin{pmatrix} 0 & w^i \\ -\bar{w}^i & 0 \end{pmatrix} u \right\} + \dots \quad (135)$$

where for brevity we denoted the scalar field  $\Phi$  by  $B^0$ . We assume that the saddle point, i.e. the distribution function and therefore the matrix  $u$ , does not depend on spin. The interaction field,  $\Phi$ , is easily integrated out, with the result that at the level of the quadratic fluctuations, the spin and charge contributions are decoupled even in the presence of the interaction.

The correction to the current is finally found as

$$\begin{aligned} \delta \mathbf{j}_b &= e\pi D N_0 \sum_{i=0}^3 \text{Re} \langle w^i \nabla_{\mathbf{x}} \bar{w}^i \rangle_{\Phi} F \\ &= 4\pi e D_{\text{qp}} \tau^2 \sum_{i=0}^3 \int d\eta dx_1 dx_2 \text{Re} \left\{ F_{t-\eta, t}(\mathbf{x}) \right. \\ &\quad \times D_{t-\eta/2, t_1-\eta/2}^{i, \eta}(\mathbf{x}, \mathbf{x}_1) F_{t_1, t_1-\eta}(\mathbf{x}_1) \\ &\quad \left. \times \tilde{\gamma}_{t_1 t_2}^i(\mathbf{x}_1, \mathbf{x}_2) (-i \nabla_{\mathbf{x}}) D_{t_2, t-\eta}^{i, \eta'=0}(\mathbf{x}_2, \mathbf{x}) \right\}, \quad (137) \end{aligned}$$

where again  $i = 0$  corresponds to the charge and  $i = 1, 2, 3$  correspond to spin channels.  $D_{tt'}^{i, \eta}(\mathbf{x}, \mathbf{x}')$  is the quasi-particle diffusion propagator in the relevant spin or charge channel, which obeys the differential equation given in equation (30), with the only difference that the diffusion constant  $D$  is replaced by the quasi-particle diffusion constant  $D_{\text{qp}}$ .  $\tilde{\gamma}^i$  is the dynamically screened interaction,

$$\tilde{\gamma}^i(\mathbf{q}, \omega) = \gamma^i [1 + \gamma^i \Pi^d / N_{\text{qp}}]^{-1} \quad (138)$$

$$= \gamma^i \frac{-i\omega + D_{\text{qp}} q^2}{-i(1 - 2\gamma^i)\omega + D_{\text{qp}} q^2}. \quad (139)$$

For the explicit calculations it is convenient to consider the product of the dynamically screened interaction and the retarded diffuson,

$$(\tilde{\gamma}^i D^i)_{tt'}(\mathbf{x}, \mathbf{x}') = \int dx_1 \tilde{\gamma}_{tt_1}^i(\mathbf{x}, \mathbf{x}_1) D_{t_1, t'}^{i, \eta=0}(\mathbf{x}_1, \mathbf{x}') \quad (140)$$

which solves the diffusion equation

$$[(1 - 2\gamma^i) \partial_t - D \nabla_{\mathbf{x}}^2] (\tilde{\gamma}^i D^i)_{tt'}(\mathbf{x}, \mathbf{x}') = \frac{\gamma^i}{\tau} \delta(\mathbf{x} - \mathbf{x}') \delta(t - t'). \quad (141)$$

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